Circumventing the Price of Anarchy
Leading Dynamics to Good Behavior

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Good equilibria, Bad equilibria

Many games have both bad and good equilibria.

• In some places, everyone drives their own car. In some, everybody uses and pays for good public transit.
Good equilibria, Bad equilibria

Fair cost-sharing

• $n$ players in directed graph $G$, each edge $e$ costs $c_e$.
• Player $i$ wants to get from $s_i$ to $t_i$.
• All players share cost of edges they use with others.
• $\text{cost}(s) = \sum_{i=1}^{n} \text{cost}_i(s)$
Good equilibria, Bad equilibria

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Good equilibrium: all use edge of cost 1.

(paying $1/n$ each)
**Good equilibria, Bad equilibria**

**Fair cost-sharing**

- \( n \) players in directed graph \( G \), each edge \( e \) costs \( c_e \).
- Player \( i \) wants to get from \( s_i \) to \( t_i \).
- All players share cost of edges they use with others.
- \( \text{cost}(s) = \sum_{i=1}^{n} \text{cost}_i(s) \)

**Good equilibrium:** all use edge of cost 1.
(paying 1/n each)

**Bad equilibrium:** all use edge of cost \( n \).
(paying 1 each)
Inefficiency of equilibria, PoA and PoS

**Price of Anarchy (PoA):** ratio of worst Nash equilibrium to OPT.  
[Koutsoupias-Papadimitriou'99]

**Price of Stability (PoS):** ratio of best Nash equilibrium to OPT.  
[Anshelevich et. al, 2004]

E.g., for fair cost-sharing, PoS is log(n), whereas PoA is n.

Significant effort spent on understanding these in CS.
Dynamics in Games

- Traditionally: convergence to some equilibria
  - Best/better response
  - Regret Minimization
  - Imitation Dynamics

- Not so satisfactory in games with a huge gap between PoA and PoS

What can we say about getting to good states?
I) Players entering one at a time

- **Undirected** single sink fair cost sharing, one at a time entering from an empty config. [Charikar et al, 2008]

- Positive result; get within polylog(n) factor of OPT

- But fails in directed graphs.
Getting to Good Equilibria

I) Players entering one at a time

• **Undirected** single sink fair cost sharing, one at a time entering from an empty config. [Charikar et al, 2008]

• Positive result; get within polylog(n) factor of OPT

Directed single sink

Bad eq. result of this dynamics:

0 0 0

1 1 1 ... 1

k ≪ n

Subway/shared van

cars

s_1

s_n
Getting to Good Equilibria

II) Noisy best response (simulated annealing on potential function)

[Blume95, Marden/Shamma08, Young05]

\[ \Pr_i(a) \propto e^{-\text{cost}_i(s_a)/\tau} \]

[Prob. of action \( a \) decreases exponentially with gap between the cost of \( a \) and cost of BR]

Show examples of directed cost sharing where no noisy-best-response alg can do better than POA within poly # of steps.

How can we get around this

Analyze if a helpful entity/source encourage (guide) behavior to move from a bad state to a good state.
Getting to Good Equilibria

III) Public Service Advertisement [BBM, SODA 2009]

- A helpful authority advertises a good joint action.
- A random constant fraction of the players follow the proposal temporarily; others do best response.

Strong positive result for fair cost sharing

If $\alpha$ fraction of players follow the advice, then get within $O(1/\alpha)$ of PoS. [PoS = $\log(n)$, PoA = $n$]

Note: The model requires:
- receptive/gullible players
- non-receptive/stubborn players.

What if each player is a bit of both?
Our Proposed Model: High level

A more adaptive model

Each player has a few **abstract actions**.
Uses a **learning, experts based alg.** to decide which one to use.

[ no rigid separation between receptive vs non-receptive players]
Our Model

Begin in some arbitrary configuration.

Someone analyzing game comes up with a good idea (joint action of low cost) and proposes it.

Players go in a random order:

With probability $p_i$ do proposed action.
With probability $1-p_i$ do best-response to current state.

**Model A:** $p_i$'s stay **fixed**, at some poly time $T^*$, everyone commits one way or the other. [Learn then Decide]

**Model B:** Players use arbitrary learning rule to slowly vary their $p_i$'s. (only limit is learning rate). [Smoothly Adaptive]

What will happen to the overall state of the system?
Our Results

Fair Cost Sharing

Learn then Decide

A poly number exploration of steps $T^*$ is sufficient s.t. the expected cost at any time $T \geq T^*$ is $O(\log(n) \log(nm)OPT)$.

Smoothly Adaptive

$n_i = \Omega(m)$, $p_i \geq \beta$ for poly steps, then $\exists T^* = \text{poly}(n)$ s.t. whp cost at any time $T \geq T^*$ is $O(\log(nm)OPT)$.

Consensus

For any graph, any initial configuration, if $p_i > \frac{1}{2}$ then whp play will reach optimal in $O(n \log^2 n)$ steps.
Fair Cost Sharing

Key Lemma #1:
So long as all $p_i \geq \epsilon$ for constant $\epsilon > 0$, whp the cost will reach $O(OPT \cdot \log(mn))$ within $poly(n)$ steps.

Proof sketch preliminaries:

- Fair cost-sharing -- exact potential game: $\exists$ potential fnc $\Phi$ s.t. if any player makes a move decreasing their own cost by $\Delta$, then $\Phi$ drops by $\Delta$ too.

$$S = (P_1, P_2, \ldots, P_n) \quad cost(S) = \sum_{e \in U_iP_i} c_e$$

$$\phi(S) = \sum e \sum_{x=1}^{n_e} f_e(x) \quad \text{where} \quad f_e(x) = \frac{c_e}{x}$$

- For any state $S$, $cost(S) \leq \Phi(S) \leq cost(S)\log(n)$. 
Fair Cost Sharing

Key Lemma #1:
So long as all $p_i \geq \epsilon$ for constant $\epsilon > 0$, whp the cost will reach $O(OPT \cdot \log(mn))$ within $\text{poly}(n)$ steps.

Proof sketch:

- After initial startup phase, whp all edges $e$ with $n_e \gg \log(nm)$ players on them in OPT, will have $\geq (\epsilon/2)n_e$ players on them now.

- Implies OPT is a “fairly good” response for everyone (cost $O(\log(nm)OPT_i)$, where $OPT_i = i$’s cost in OPT).

- So, if cost is currently high, if player $i$ picked at random, expected drop in $\Phi$ is large (whether $i$ does proposed action or BR).

- Can’t happen for too long (use martingale tail bound).
**Fair Cost Sharing**

**Key Lemma #1:**
So long as all $p_i \geq \epsilon$ for constant $\epsilon > 0$, whp the cost will reach $O(OPT \cdot \log(mn))$ within $\text{poly}(n)$ steps.

Great - cost gets low pretty soon!

But not quite enough to get what we want... need to ensure don't have:
Fair Cost Sharing, Learn then Decide

Key Lemma #1:
So long as all \( p_i \geq \epsilon \) for constant \( \epsilon > 0 \), whp the cost will reach \( O(OPT \cdot \log(mn)) \) within \( \text{poly}(n) \) steps.

Key Lemma #2:
So long as all \( p_i \geq \epsilon \) for constant \( \epsilon > 0 \), if cost at time \( T_1 \) is \( O(OPT \cdot \log(mn)) \), then \( E[\Phi] \) at any time \( T = T_1 + \text{poly}(n) \) is \( O(OPT \cdot \log(mn) \cdot \log(n)) \).

Final step for learn then decide model:
In final decision step, potential cannot increase by much.
Fair Cost Sharing, Smoothly Adaptive

**Key Lemma #1:**
So long as all $p_i \geq \epsilon$ for constant $\epsilon > 0$, whp the cost will reach $O(OPT \cdot \log(mn))$ within $\text{poly}(n)$ steps.

**Final step for adaptive model:**
If *many players of each type* can show that once cost is low, it will *never* get high again.
Fair Cost Sharing, Smoothly Adaptive

Final step for adaptive model:
If *many players of each type* can show that once cost is low, it will *never* get high again.

Proof sketch:
Say cost is low at time $t_0$.

- Hard to analyze cost of state directly, instead track upper bound $c^*(S_t) = \text{cost}(S_{t_0} \cup ... \cup S_t)$.
- $c^*$ changes at most $m$ times.
- Many players of each type $\Rightarrow$ average cost of each is low compared to $c^*$. Each change to $c^*$ is small. $(c^*/n_i)$

Total cost ever at most: $\text{cost}(S_0)(1 + 1/n_{min})^m$
Consensus games

- Graph $G$, vertices have two actions: RED or BLUE.

  \[
  \text{cost}_i(s) = \sum_{(i, j) \in E} I_{s_i \neq s_j}
  \]

  Pay 1 for each edge with endpoints of different color.

  \[
  \text{cost}(s) = \sum_i \text{cost}_i(s) + 1
  \]

- OPT = all RED or all BLUE. Cost(OPT) = 1.
Consensus games

• OPT is an equilibrium so \( \text{PoS} = 1 \). But \( \text{PoA} = \Omega(n^2) \).

Two cliques of size \( n \). Each node has \( \epsilon n \) nbrs in other clique, \( \epsilon < 1 \).

• In fact, the bad equilibrium can be pretty stable.

• Even if proposal = “all BLUE”, for any \( p < \frac{1}{2} \), if \( \epsilon < \frac{1}{2} - p \) then whp BR is to keep orig color and so no change....
Consensus games

Main result:

For any graph, any initial configuration, if \( p > \frac{1}{2} \), then whp play will reach optimal in \( O(n \log^2 n) \) steps. \([\text{proposal} = \text{all BLUE}]\)

Main idea:

- Two ways a node can become blue: by choosing the proposed action or because it has more blue neighbors than red neigh, so BR is blue
- Even if many dependencies among neighs, \( \Pr(\text{BR is blue}) \) increases quickly over time.
Conclusions

Propose a novel perspective for leading dynamics to a good equilibrium and get around inherent lower bounds.

- Analyze process where some entity (who studies the game and discovers a good behavior) proposes a good joint action.

- Players don’t trust, so view proposal and best-response as two “experts” and run arbitrary learning alg between them

- Positive results for cost-sharing and consensus games.
Open Questions

- Remove restriction on many players of each type for adaptive model.
- Extend model to allow multiple proposed actions, hope to do (nearly) as well as the best.
- Alternative ways to give players more info about game they are playing to allow them to reach good states fast?