Bounding Rationality by Discounting Time

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Plan of the Talk

• Introduction
• The Model
• Results
• Future Directions
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Perfect Rationality

• Perfect rationality in a strategic situation
  – Each player is rational (knows its payoff, and wishes to maximize it)
  – Each player knows that the other player is rational
  – Each player can derive all consequences of common rationality
Bounded Rationality

- Herbert Simon – “Boundedly rational agents experience limits in formulating and solving complex problems and in processing (receiving, storing, retrieving, transmitting) information”
- In particular, boundedly rational agents are subject to computational constraints
Games

• Simultaneous-move (eg., Prisoner’s Dilemma) or Sequential-move (eg., chess)

• Simultaneous-move
  – Action spaces: $A_1, A_2$
  – Strategy spaces: $P(A_1), P(A_2)$
  – Payoff functions: $A_1 \times A_2 \rightarrow \mathbb{R}$

• Sequential-move (one-shot)
  – Strategy spaces: $P(A_1), P(A_2)^{^\text{A}_1}$
  – Payoff functions: $A_1 \times A_2 \rightarrow \mathbb{R}$
Nash Equilibrium

• A pair of strategies \((S_1, S_2)\) is an NE if
  – For all \(T_2\), \(u_2(S_1, S_2) \geq u_2(S_1, T_2)\)
  – For all \(T_1\), \(u_1(S_1, S_2) \geq u_1(T_1, S_2)\)

• Theorem [Nash]: Every finite game has an NE
Almost-Nash Equilibrium

• A pair of strategies \((S_1, S_2)\) is a \(\gamma\)-NE if
  – For all \(T_2\), \(u_2(S_1, S_2) \geq u_2(S_1, T_2) - \gamma\)
  – For all \(T_1\), \(u_1(S_1, S_2) \geq u_1(T_1, S_2) - \gamma\)
# The Largest Number Game

<table>
<thead>
<tr>
<th>Alice</th>
<th>Bob</th>
<th>Payoff (to Alice)</th>
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<tbody>
<tr>
<td>$M$ (Integer)</td>
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<td>100 if $M &gt; N$, 50 if $M = N$, 0 otherwise</td>
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Largest Number game does not have an NE, or even an almost-NE if $\gamma < 50$
The Factoring Game (sequential-move)

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Factoring Game has infinitely many Nash equilibria, in each of which Bob gets payoff 100 and Alice gets payoff 1 (Bob’s strategy is simply to factor Alice’s number).
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Time is Money

- The *time* it takes to implement a strategy is relevant
- Payoffs should decrease with time
- Exponential discounting: Let $\varepsilon < 1$ be a discount factor. Then payoff decreases by a factor $(1-\varepsilon)^t$ after $t$ steps
Asymmetric Discounting

• In general, different players have different discount factors
  – The players might have different roles in the game
  – Even if the game is symmetric, the players themselves might not be equally patient

• $\varepsilon$: Alice’s discount factor

• $\delta$: Bob’s
Discounting and Computational Power

- By “time” we mean *computational time*
- Suppose Alice and Bob are equally patient with respect to real time but Alice’s computer is 100 times as powerful as Bob’s. Then $\delta \sim 100 \, \varepsilon$
- Discount factor is not just an index of patience, but also of computational power
The Discounted Game

• Let $G = (A_1, A_2, u_1, u_2)$ be a game
• The $(\varepsilon, \delta)$-discounted version of $G$ has
  – Actions: Probabilistic machines which take as input $\varepsilon$ and $\delta$, and output actions in $A_1$ (resp. $A_2$)
  – Payoffs: Alice’s payoff corresponding to machines $M_1$ (Alice) and $M_2$ (Bob) outputting $a_1 \in A_1$ and $a_2 \in A_2$ resp. is $u_1(a_1,a_2)(1 - \varepsilon)^t$, where $t$ is time taken for $M_1$ to output $a_1$
Uniform Equilibria

- A pair of strategies \((S_1, S_2)\) for the discounted game is a uniform NE if neither player can gain in the limit as \(\varepsilon, \delta \to 0\) by playing a different strategy.

- Limit case interesting because
  - \(\varepsilon, \delta\) are typically small
  - As computational power increases, \(\varepsilon\) and \(\delta\) get smaller
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Finite Games

• Theorem: Let G be a finite game. For every NE of G, the discounted version of G has a uniform NE with the same payoffs in the limit.
Infinite Games

• Theorem: Every countable game with bounded computable payoffs has a uniform NE

• Note that such games do not always have an NE or even an almost-NE (e.g., Largest Number Game)
The Largest Number Game, Revisited

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The Largest Number Game, Revisited

• All the uniform equilibria of Largest Number game yield payoff 0 for both players
• Example: both players play $2^{1/\varepsilon^2 + 1/\delta^2}$
• If more is known about relationship between $\varepsilon$ and $\delta$, eg., $\varepsilon \gg \delta$, then there might be other equilibria yielding non-zero payoffs
The Factoring Game, Revisited

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Factoring Game has infinitely many Nash equilibria, in each of which Bob gets payoff 100 and Alice gets payoff 1 (Bob’s strategy is simply to factor Alice’s number)
Complexity Through Game Theory

• Tight connection between computational complexity of Factoring and uniform equilibrium payoffs of discounted Factoring game

• Let $\delta = \epsilon^c$, for some $c > 1$, wlog

• Theorem: If Factoring is in time $o(n^c)$ on average, then every uniform NE of discounted game gives payoff 1 to Alice and 100 to Bob
Complexity Through Game Theory

• Theorem: Suppose there is no algorithm which runs in time $n^c \cdot \text{polylog}(n)$ and solves Factoring on average for infinitely many input lengths. Then there is a uniform NE of discounted game giving payoff 100 to Alice and 1 to Bob.

• Proof idea: Consider strategy for Alice of outputting random number of size $\sim 1/\epsilon$. Show that any strategy for Bob yielding payoff more than 1 in the limit yields factoring algorithm
A Spurious Equilibrium

• In the case where Factoring is hard, there is still a uniform NE where Bob wins
• This corresponds to Bob playing a brute-force Factoring algorithm
• However, in practice, we wouldn’t expect this to happen – Bob’s threat is not credible
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Future Directions: Refining the Model

• Defining a notion of subgame-perfection for discounted games
• An approach based on preference relations rather than real-number payoffs
• Capture bounded rationality not just in implementation but also in design
Future Directions: Applications of the Model

- Using discounting in choice situations ("flexible" or "anytime" algorithms)
- Perspective on foundations of cryptography, where protocol is treated as a game and adversary is modelled as bounded-rational
- Bounded rationality in extensive-form games