Analytical Tools for Natural Algorithms

Bernard Chazelle
Princeton University
three multiagent agreement systems
Hegselmann-Krause opinion dynamics

$n$ opinions
each opinion averages itself with similar opinions
figure by Urbig-Lorenz-Herzberg
Kuramoto sync
all of these dynamical systems converge in time

\[ 2^{O(n)} \]

where \( n \) is the number of opinions, metronomes, fireflies, etc.
no previous convergence bound was known
let’s step back a little
$P^t_x$

predict behavior
for large $t$
spectral decomposition

\[ P^t x \]

dimension separation

independent 1-dim systems
that was was easy
what about?
\[ P_t \cdots P_2 P_1 x \]

where

\[ P_t = f(x, P_1, \cdots, P_{t-1}) \]

nonlinear dynamics
hopeless
attack

\[ P_{t_2} \ldots P_{t_1} x \]

as

\[ P_t(\ldots(P_{2}(P_{1}x))) \]
bye-bye old math

hello data analysis
classification
machine learning
statistics
???

$P_t$
good old math

rarely works
data analysis

algorithm

works if you redefine the meaning of “works”
relativity theory
relativity theory via data analysis
relativity theory via data analysis

\[ E = m^{0.92} c^{2.03} \]
algorithm
use algorithms to analyze algorithms
back to our 3 agreement systems
a common geometric framework
$n$ points in $E^d$
infinite graph sequence
shrink by factor $1 - \rho$
move vertex anywhere
repeat for each node
get second graph
move all the vertices
get 3rd graph, move vertices, repeat...
adversary chooses all graphs
and all moves nondeterministically
does this thing always converge?
yes
convergence

radius $\varepsilon$
Angeli, Bliman, Blondel, Cao, Cucker, Hendrickx, Jadbabaie, Lin, Moreau, Morse, Olshevsky, Spielman, Tsitsiklis, Wang, etc.

necessary conditions
no convergence time
THEOREM 1

System always converges:

(i) number of moves is infinite

(ii) number of nontrivial moves

$$\leq \min \left\{ \varepsilon^{-1} \rho^{-O(n)} , \ (\log 1/\varepsilon)^{n-1} \rho^{-n^2(1+o(1))} \right\}$$
| main tool | total s-energy |
$$E(s) = \sum_t \sum_{(i,j) \in G_t} \| x_i - x_j \|_2^s$$
$E(s)$

Dirichlet series $\rightarrow$ inverse formula

(Mellin transform) $\rightarrow$ lossless encoding

Does it ever converge?
$E(s)$

- converges for all $s > 0$
- analytic for $\Re s > 0$
- pole at $s = 0$ of order $n - 1$
in general, no analytic continuation over whole plane

conjecture

for any $n$, $\max E(s)$ has analytic continuation with discrete poles at $s=0$
true for $n = 2$
THEOREM 2

\[ E(s) \leq \begin{cases} 
\rho^{O(n)} & \text{if } s = 1 \\
S^{n-1} \rho^{-n^2(1+o(1))} & \text{if } s < 1 
\end{cases} \]

proof: algorithmicized proofs of old math

flow algorithm
recurrences, etc.
\[ P = u^{-1} T u \]

Schur’s Lemma

Every square matrix is unitarily similar to a triangular matrix
Unitary equivalence and normal matrices

of \( C^n \). Apply the Gram–Schmidt orthonormalization procedure to this basis to produce an orthonormal basis

\[ x^{(1)}, x^{(2)}, \ldots, x^{(n)} \]

of \( C^n \). Arrange these orthonormal vectors left to right as the columns of a unitary matrix \( U \). Since the first column of \( AU \) is \( \lambda x^{(1)} \), a calculation reveals that \( U^*AU \) has the form

\[ U_A^*AU = \begin{bmatrix}
\lambda_1 & * \\
0 & \bar{A}_t
\end{bmatrix} \]

The matrix \( A_t \in M_{n-1} \) has eigenvalues \( \lambda_2, \ldots, \lambda_n \). Let

\[ x^{(2)} = \frac{x^{(2)}}{\|x^{(2)}\|} \]

normalized eigenvector of \( A_t \) corresponding to \( \lambda_2 \), and do it all over again. Determine a unitary \( U_2 \in M_{n-1} \) such that

\[ U_2^*A_tU_2 = \begin{bmatrix}
\lambda_2 & * \\
0 & \bar{A}_2
\end{bmatrix} \]

and let

\[ V_2 = \begin{bmatrix}
1 & 0 \\
0 & U_2
\end{bmatrix} \]

The matrices \( V_2 \) and \( U_1V_2 \) are then unitary, and \( V_2^*U_1^*AU \) has the form:

\[ V_2^*U_1^*AU_1V_2 = \begin{bmatrix}
\lambda_1 & * \\
0 & \lambda_2 \\
0 & \bar{A}_2
\end{bmatrix} \]

Continue this reduction to produce unitary matrices \( U_i \in M_{n-i+1}, i = 1, 2, \ldots, n-1 \) and unitary matrices \( V_i \in M_{n-i}, i = 2, \ldots, n-1 \). The matrix

\[ U = U_1V_2V_3 \ldots V_{n-1} \]

is unitary and \( U^*AU \) yields the desired form.

If all eigenvalues of \( A \in M_n(\mathbb{R}) \) happen to be real, then the corresponding eigenvectors can be chosen to be real and all the above steps may be carried out in real arithmetic, verifying the final assertion.

Remark: Follow the proof of (2.3.1) to see that “upper triangular” could be replaced by “lower triangular” in the statement of the theorem, of course, a different unitary equivalence \( U \).

2. Example. Neither the unitary matrix \( U \) nor the triangular matrix. 

Theorem (2.3.1) is unique. Not only may the diagonal entries of \( T \). 

2.3 Schur’s unitary triangularization theorem

The eigenvalues of \( A \) appear in any order, but unitarily equivalent upper triangular matrices may appear very different above the diagonal. For example,

\[ T_1 = \begin{bmatrix}
1 & 1 & 4 \\
0 & 2 & 2 \\
0 & 0 & 3
\end{bmatrix} \quad \text{and} \quad T_2 = \begin{bmatrix}
2 & -1 & 3\sqrt{2} \\
0 & 1 & \sqrt{2} \\
0 & 0 & 3
\end{bmatrix} \]

are unitarily equivalent via

\[ U = \frac{1}{\sqrt{2}} \begin{bmatrix}
1 & 1 & 0 \\
1 & -1 & 0 \\
0 & 0 & \sqrt{2}
\end{bmatrix} \]

In general, many different upper triangular matrices can be in the same unitary equivalence class.

Remark: Notice that the technique of the proof (2.3.1) is simply that of sequential deflation, as outlined in Problem 8 in Section 1.4.

Exercise. If \( A \in M_n \) is unitarily equivalent to an upper triangular matrix, the entries \( t_{ij} \) are not uniquely determined, but the quantity \( \sum_{t_{ij}} |t_{ij}|^2 \) is uniquely determined. Determine the value of \( \sum_{t_{ij}} |t_{ij}|^2 \) in terms of the entries and eigenvalues of \( A \). Hint: Use (2.2.2).

Exercise. If \( A = \{a_{ij}\} \) and \( B = \{b_{ij}\} \in M_2 \) are similar and if \( \sum_{i,j} |a_{ij}|^2 = \sum_{i,j} |b_{ij}|^2 \), show that \( A \) and \( B \) are unitarily equivalent. Show by example that this is not the case in higher dimensions. Hint: Notice that if \( A \) and \( B \) are unitarily equivalent, then so are \( A + A^* \) and \( B + B^* \). Consider

\[ A = \begin{bmatrix}
1 & 3 & 0 \\
0 & 2 & 4 \\
0 & 0 & 3
\end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix}
1 & 0 & 0 \\
0 & 2 & 5 \\
0 & 0 & 3
\end{bmatrix} \]

It is a useful adjunct to (2.3.1) that a commuting family of matrices may be simultaneously upper triangularized.

2.3.3 Theorem. Let \( \mathcal{F} \subseteq M_n \) be a commuting family. There is a unitary matrix \( U \in M_n \) such that \( U^*AU \) is upper triangular for every \( A \in \mathcal{F} \).

Proof: Return to the proof of (2.3.1). Exploiting (1.3.17) at each step of the proof in which a choice of an eigenvector (and unitary matrix) is made, the same eigenvector (and unitary matrix) may be chosen for every \( A \in \mathcal{F} \). Moreover, unitary equivalence preserves commutativity.
proof is algorithmic
proof is algorithmic

flow algorithm

set up recurrence relations

extract algorithm

modify it

convergence
use algorithms to analyze algorithms
and stay warm